On the 5/8 bound for non-Abelian Groups

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Abstract

If we pick two elements of a non-abelian group at random, the odds this pair commutes is at most 5/8, so there is a "gap" between abelian and non-abelian groups [6]. We prove a "topological" genearlization estimating the odds a word representing the fundamental group of an orientable surface $\langle x, y : [x_1, y_1][x_2, y_2] \dots [x_n, y_n] = 1 \rangle$ is satisfied. This resolve a conjecture by Langley, Levitt and Rower.

1 Counting Solutions to Equations in Groups

Kopp and Wiltshire-Gordon wanted to know how often a given word, $w = w_1 w_2 \dots w_n$ is satisfied by elements of a group [2].

$$\gamma_G(w) = \#\{(g_1, \dots, g_n) \in G^n : w(g) = 1\}$$

Equivalently, they define a measure μ_w and for functions $f : G \to \mathbb{C}$

$$\int_G f d\mu_w := \int_{G^n} f(w(g_1, \dots g_n)) dg_1 \dots dg_n$$

Every word has an associated 2-dimensional CW complex, e.g. the commutator $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ corresponds to the torus, $\mathbb{T}^2$ and $g_1 g_2 g_3 g_1^{-1} g_4 g_3^{-1} g_2^{-1} g_4^{-1}$ is also a torus.

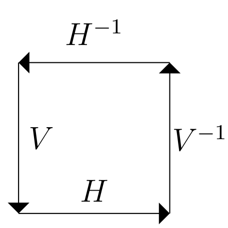

Figure 1: The torus corresponds to the word $[V, H] = VHV^{-1}H^{-1}$.

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For any word w, let $X(w)$ be the CW complex associated to this word.

Theorem 1. *[2] Let w_1, w_2 be words and G be compact group. If $X(w_1) \simeq X(w_2)$ are homeomorphic,*

$$\int_{G^m} f(w_1(\vec{g}))d\vec{g} = \int_{G^n} f(w_2(\vec{g}))d\vec{g}$$

for all measurable functions $f : G \to \mathbb{C}$. This means $\mu_{w_1} = \mu_{w_2}$.

In other words, they establish the measures μ_w are invarants of surfaces. This can be proven using 2D Yang-Mills theory [1, 3, 4] or non-abelian group cohomology[1]. They also calculate the measure for a connected sum of tori and hence for all orientable surfaces.

Theorem 2. *[2] Let w be a word defining a orientable surface (i.e. having each g_k and g_k^{-1} appear only once) and let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. The average value of $\rho(w)$ is proportional to the identity.*

$$\int_{G^n} \rho(w(\vec{g}))\, d\vec{g} = (\dim V)^{k-2} I$$

Taking the trace of both sides

$$\int_{G^n} \rho(w(\vec{g}))d\vec{g} = (\dim V)^{k-1}$$

For any Haar-measurable function $f : G \to \mathbb{C}$

$$\int_{G^n} f(w(\vec{g}))\, d\vec{g} = \sum_{\rho\in\hat{G}} \langle \rho | f \rangle (\dim \rho)^{k-1}$$

Here the inner product $\langle \rho | f \rangle = \int_G \chi_\rho(g)\overline{f(g)}\, dg$ integrated with respect to Haar measure.

1.1 5/8 Bound

The word $ghg^{-1}h^{-1} = 1$ corresponds to the torus if we label the two homology cycles with group elements g, h.

Theorem 3. *Let G be a nonabelian group.*

$$\frac{\{(g,h) \in G^2 : ghg^{-1}h^{-1} = 1\}}{|G|^2} = \frac{c(G)}{|G|} \leq \frac{5}{8}$$

where $c(G)$ is the number of conjugacy classes.

[1] http://terrytao.wordpress.com/2012/05/11/cayley-graphs-and-the-algebra-of-groups/

Proof by Will Sawin. [2] Assume $c(G) > \frac{5}{8}|G|$. The order of the group is a sum over all characters

$$|G| = \sum (\dim \rho)^2$$

and remember there are as many irreducible representations as conjugacy classes. Get something like

$$\frac{8}{5}\langle 1 \rangle |G| = \frac{8}{5}\sum_{\rho} 1 > \sum_{\rho} (\dim \rho)^2 = \langle (\dim \rho)^2 \rangle |G|$$

On average the dimension-squared of the character $(\dim \rho)^2$ is less than $8/5$. Then let x be the fraction of representations with dimension at least 1 and $(1-x)$ of them have dimension-squared at least 4.

$$x \cdot 1 + (1-x) \cdot 4 < \frac{8}{5} \quad \text{so that} \quad x > \frac{4}{5} \quad \text{fraction are 1D characters}$$

Every homomorphism $\phi : G \to \mathbb{C}$ should satisfy $\phi(gh) = \phi(g)\phi(h) = \phi(h)\phi(g) = \phi(hg)$, so it is well-defined on the quotient $G/[G,G]$. The abelianization $G/[G,G]$ will have one element per 1-dimensional character of G.

$$|G/[G,G]| = \frac{|G|}{|[G,G]|} > \frac{4}{5}|G| \quad \text{so that} \quad |[G,G]| < \frac{5}{4}$$

and so $|[G,G]| = 1$, every pair of elements commute. ♣

Instead of using the commutator word $ghg^{-1}h^{-1}$ we could use any word corresponding to a surface (since we have a bound for it). Let's check a conjecture by Langley, Levitt and Rower bounding the probability and word is equal to its rearrangement, [5].

Theorem 4. *For any non-abelian group, $n \geq 2$ and $\sigma \in S_n$,*

$$\frac{\#\{(a_1, \dots, a_n) : a_1 \dots a_n = a_{\sigma(1)} \dots a_{\sigma(n)}\}}{|G|^n} \leq \frac{1}{2} + \frac{1}{2^{2k+1}}$$

where k is the fewest number of block transpositions in a factorization of σ.

In other words, if the odds of a given word being satisfied is too much past 50%, the group must be abelian.

A block transposition transposes two disjoint blocks of consecutive elements.

$$(a_1 a_2 a_3 a_4 a_5)^{(1,3,5,2,4)} = (a_4 a_5)(a_1 a_2 a_3)$$

transposing the two blocks $[1,2,3]$ and $[4,5]$. Topologically, if we consider the word $a_1 a_2 \dots a_n (a_{\sigma(1)} a_{\sigma(2)} \dots a_{\sigma(n)})^{-1}$ the minimum number of block transpositions k is the genus of the surface.

[2]http://mathoverflow.net/questions/91685/5-8-bound-in-group-theory

Proof by Contradiction. The number of permutations satisfying the word $w = a_1 a_2 \dots a_n (a_{\sigma(1)} a_{\sigma(2)} \dots a_{\sigma(n)})^{-1}$ can be computed exactly.

$$\frac{\#\{(a_1, \dots, a_n) : a_1 \dots a_n = a_{\sigma(1)} \dots a_{\sigma(n)}\}}{|G|^n} = \sum_{\rho \in \mathrm{Irr}(G)} (\dim \rho)^{k-1}$$

Assume to the contrary that

$$\left(\frac{1}{2} + \frac{1}{2^{2k+1}}\right) \sum_{\rho \in \mathrm{Irr}(G)} (\dim \rho)^{k-1} > \sum_{\rho \in \mathrm{Irr}(G)} (\dim \rho)^2$$

Let $x = 1/|[G,G]|$ be the fraction of characters that are Abelian. Then next lowest dimension is $\dim \rho = 2$.

$$\begin{aligned}
\left(\frac{1}{2} + \frac{1}{2^{2k+1}}\right)(x \cdot 1 + (1-x) \cdot 2^{k-1}) &> (x \cdot 1 + (1-x) \cdot 4) \\
\left(\left(\frac{1}{2} + \frac{1}{2^{2k+1}}\right)(1 - 2^{k-1}) + 3\right) x &> 4 - \left(\frac{1}{2} + \frac{1}{2^{2k+1}}\right) 2^{k-1} \\
\frac{1}{|[G,G]|} = x &> \frac{2^{k-1} - \frac{4}{\frac{1}{2} + \frac{1}{2^{2k+1}}}}{2^{k-1} - 1 - \frac{3}{\frac{1}{2} + \frac{1}{2^{2k+1}}}} > 1
\end{aligned}$$

This is a contradiction since $[G,G]$ contains the identity so $|[G,G]| \geq 1$. ♣

Proof by example. Our proof implicitly uses some non-group cohomology when we cite the Midgal formula or in order to show the probably measure on $w : G^n \to \mathbb{R}$ is a topological invariant.

In our example $g_1 g_2 g_3 g_1^{-1} g_4 g_3^{-1} g_2^{-1} g_4^{-1}$, we can draw an octagon with some sides identified and label the edges with the group elements. Looking at the diagram we can rearrange our

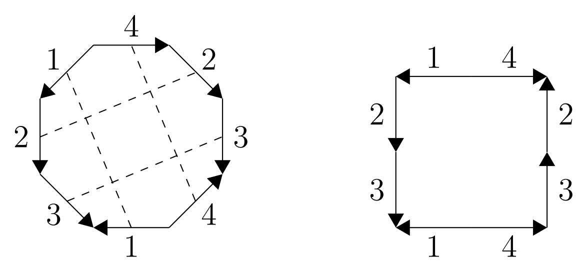

Figure 2: The surface corresponding to the word $1231^{-1}43^{-1}2^{-1}4^{-1}$ is a torus.

word into the product of commutators.

$$\mathbb{P}\left(g_1 g_2 g_3 g_1^{-1} g_4 g_3^{-1} g_2^{-1} g_4^{-1} = 1\right) = \mathbb{P}\left([g_2 g_3, g_4^{-1} g_1] = 1\right) = \mathbb{P}([g,h] = 1)$$

The products $g = g_2 g_3, h = g_4^{-1} g_1$ will be uniformly random so we gave them new variable names. From the pictures or the algebra, it's clear there was only a single block transposition and so this diagram is a genus 1 surface. This word should have the same statistics as $[g,h] = ghg^{-1}h$

Establishing the measures the same, the 5/8 bound must hold here as well. ♣

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